Qubits from Adinkra Graph Theory via Colored Toric Geometry

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Abstract

We develop a new approach to deal with qubit information systems using toric geometry and its relation to Adinkra graph theory. More precisely, we link three different subjects namely toric geometry, Adinkras and quantum information theory. This one to one correspondence may be explored to attack qubit system problems using geometry considered as a powerful tool to understand modern physics including string theory. Concretely, we examine in some details the cases of one, two, and three qubits, and we find that they are associated with $\mathbf{CP^1}$, $\mathbf{CP^1 \times CP^1}$ and $\mathbf{CP^1 \times CP^1 \times CP^1}$ toric varieties respectively. Using a geometric procedure referred to as colored toric geometry, we show that the qubit physics can be converted into a scenario handling toric data of such manifolds by help of Adinkra graph theory. Operations on toric information can produce universal quantum gates.

Keywords: Toric geometry; information theory and Adinkra graph theory.

1 Introduction

Toric geometry is considered as a nice tool to study complex varieties used in physics including string theory and related models[1, 2]. The key point of this method is that the geometric properties of such manifolds are encoded in toric data placed on a polytope consisting of vertices linked by edges. The vertices satisfy toric constraint equations which have been explored to solve many string theory problems such as the absence of non abelian gauge symmetries in ten dimensional type II superstring spectrums[3].

Moreover, toric geometry has been also used to build mirror manifolds providing an excellent way to understand the extension of T-duality in the presence of D-branes moving near toric Calabi-Yau singularities using combinatorial calculations [4]. In particular, these manifolds have been used in the context of $N = 2$ four dimensional quantum field theories in order to obtain exact results using local mirror symmetry[3]. Besides such applications, toric geometry has been also explored to understand a class of black hole solutions obtained from type II superstrings on local Calabi-Yau manifolds [5, 6].

Recently, the black hole physics has found a place in quantum information theory using qubit building blocks. More precisely, many connections have been established including the link with STU black holes as proposed in [7, 8, 9].

More recently, an extension to extremal black branes derived from the T^n toroidal compactification of type IIA superstring have been proposed in [10]. Concretely, it has been shown that the corresponding physics can be related to n qubit systems via the real Hodge diagram of such compact manifolds. The analysis has been adopted to $T^{n|n}$ supermanifolds by supplementing fermionic coordinates associated with the superqubit formalism and its relation to supersymmetric models.

The aim of this paper is to contribute to this program by introducing colored toric geometry and its relation to Adinkra graph theory to approach qubit information systems. An objective here is to connect three different subjects namely toric geometry, Adinkras and quantum information theory. This link could be explored to deal with qubit systems using geometry considered as a powerful tool to understand modern physics. As an illustration, we examine lower dimensional qubit systems. In particular, we consider in some details the cases of one, two and three qubits, we find that they are linked with $\mathbf{CP^1}$, $\mathbf{CP^1} \times \mathbf{CP^1}$ and $\mathbf{CP^1} \times \mathbf{CP^1} \times \mathbf{CP^1}$ toric varieties respectively. Using a geometric procedure referred to as colored toric geometry, we show that the qubit physics can be converted into a scenario working with toric data of such manifolds by help of Adinkra graph theory.

The present paper is organized as follows. Section 2 provides materials on how colored toric geometry may be used to discuss qubit information systems. The connection with Adinkra graph theory is investigated in Section 3 where focus is on an one to one correspondence

between Adinkras, colored toric geometry and qubit systems. Operations on toric graphs are employed in Section 4 when studying universal quantum gates. Section 5 is devoted to some concluding remarks.

2   Colored toric geometry and Adinkras

Before giving a colored toric realization of qubit systems, we present an overview on ordinary toric geometry. It has been realized that such a geometry is considered as a powerful tool to deal with complex Calabi-Yau manifolds used in the string theory compactification and related subjects[2]. Many examples have been elaborated in recent years producing non trivial geometries.

Roughly speaking, n-complex dimensional toric manifold, which we denote as $M_\triangle^n$, is obtained by considering the $(n + r)$-dimensional complex spaces C^{n+r}, parameterized by homogeneous coordinates $\{x = (x_1, x_2, x_3, ..., x_{n+r})\}$, and r toric transformations T_a acting on the x_i's as follows

$$T_a : x_i \rightarrow x_i \left(\lambda_a^{q_i^a} \right). \tag{2.1}$$

Here, λ_a's are r non vanishing complex parameters. For each a, q_i^a are integers which called Mori vectors encoding many geometrical information on the manifold and its applications to string theory physics. In fact, these toric manifolds can be identified with the coset space C^{n+r}/C^{*r}. In this way, the nice feature is the toric graphic realization. Concretely, this realization is generally represented by an integral polytope Δ, namely a toric diagram, spanned by $(n + r)$ vertices v_i of the standard lattice Z^n. The toric data $\{v_i, q_i^a\}$ should satisfy the following r relations

$$\sum_{i=0}^{n+r-1} q_i^a v_i = 0, \qquad a = 1, \ldots, r. \tag{2.2}$$

Thus, these equations encode geometric data of $M_\triangle^n$. In connection with lower dimensional field theory, it is worth noting that the q_i^a integers are interpreted, in the $\mathcal{N} = 2$ gauged linear sigma model language, as the $U(1)^r$ gauge charges of $\mathcal{N} = 2$ chiral multiples. Moreover, they have also a nice geometric interpretation in terms of the intersections of complex curves C_a and divisors D_i of $M_\triangle^n$ [3, 4, 11]. This remarkable link has been explored in many places in physics. In particular, it has been used to build type IIA local geometry.

The simplest example in toric geometry, playing a primordial role in the building block of higher dimensional toric varieties, is $\mathbf{CP^1}$. It is defined by $r = 1$ and the Mori vector charge takes the values $q_i = (1, 1)$. This geometry has an U(1) toric action $\mathbf{CP^1}$ acting as follows

$$z \rightarrow e^{i\theta} z, \tag{2.3}$$

where $z = \frac{x_1}{x_2}$, with two fixed points v_0 and v_1 placed on the real line. The latters describing the North and south poles respectively of such a geometry, considered as the (real) two-sphere $\mathbf{S}^2 \sim \mathbf{CP^1}$, satisfy the following constraint toric equation

$$v_0 + v_1 = 0. \tag{2.4}$$

In toric geometry language, $\mathbf{CP^1}$ is represented by a toric graph identified with an interval $[v_0, v_1]$ with a circle on top. The latter vanishes at the end points v_0 and v_1. This toric representation can be easily extended to the n-dimensional case using different ways. The natural one is the projective space $\mathbf{CP^n}$. In this way, the $\mathbf{S^1}$ circle fibration, of $\mathbf{CP^1}$, will be replaced by $\mathbf{T}^n$ fibration over an n-dimensional simplex (regular polytope). In fact, the $\mathbf{T}^n$ collapses to a $\mathbf{T}^{n-1}$ on each of the n faces of the simplex, and to a $\mathbf{T}^{n-2}$ on each of the $(n-2)$-dimensional intersections of these faces, etc.

The second way is to consider a class of toric varieties that we are interested in here given by a trivial product of one dimensional projective spaces $\mathbf{CP^1}$'s admitting a similar description. We will show later on that this class can be used to elaborate a graphic representation of quantum information systems using ideas inspired by Adinkra graph theory and related issues [12-18]. For simplicity reason, we deal with the case of $\mathbf{CP^1} \times \mathbf{CP^1}$. For higher dimensional geometries $\bigotimes_{i=1}^{n} \mathbf{CP_i^1}$ and their blow ups, the toric descriptions can be obtained using a similar way. In fact, they are n dimensional toric manifolds exhibiting $U(1)^n$ toric actions. A close inspection shows that there is a similarity between toric graphs of such manifolds and qubit systems using a link with Adinkra graph theory. To make contact with quantum systems, we reconsider the study of toric geometry by implementing a new toric data associated with the color, producing a colored toric geometry. In this scenario, the toric data $\{v_i, q_i^a\}$ will be replaced by

$$\{v_i, q_i^a, c_j, \quad j = 1, \ldots n\}. \tag{2.5}$$

where c indicates the color of the edges linking the vertices. Roughly speaking, the connection that we are after requires that the toric graph should consist of $n + r$ vertices and n colors. In fact, consider a special class of toric manifolds associated with $\bigotimes_{i=1}^{n} CP_i^1$ with $U(1)^n$ toric actions exhibiting 2^n fixed points v_i. In toric geometry langauge, the manifolds are represented by 2^n vertices v_i belonging to the Z^n lattice satisfying n toric equations. It is observed that these graphs share a strong resemblance with a particular class of Adinkras formed by 2^n nodes connected with n colored edges [9]. These types of graphs are called regular ones which can be used to present graphically the n-qubit systems. At first sight, the connection is not obvious. However, our main argument is based on the Betti number calculations. In fact, these numbers b_i appear in Adinkras and the corresponding toric graphs. For $\mathbf{CP^1} \times \mathbf{CP^1}$, it is easy to calculate such numbers. They are given by

$$b_0 = 1, \qquad b_2 = 2, \qquad b_4 = 1 \tag{2.6}$$

Indeed, these numbers can be identified with $(1, 2, 1)$ data used in the $n = 2$ classification of Adinkras.

3   Andinkras and colored toric geometry of qubits

Inspired by combinatorial computations in quantum physics, we explore colored toric geometry to deal with qubit information systems [19-29]. Concretely, we elaborate a toric description in terms of a trivial fibration of one dimensional projective space $\mathbf{CP^1}$'s. To start, it is recalled that the qubit is a two state system which can be realized, for instance, by a $1/2$ spin atom. The superposition state of a single qubit is generally given by the following Dirac notation

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \tag{3.1}$$

where a_i are complex numbers satisfying the normalization condition

$$|a_0|^2 + |a_1|^2 = 1. \tag{3.2}$$

It is remarked that this constraint can be interpreted geometrically in terms of the so called Bloch sphere, identified with $\frac{SU(2)}{U(1)}$ quotient Lie group [1, 2, 3, 4]. The analysis can be extended to more than one qubit which has been used to discuss entangled states. In fact, the two qubits are four quantum level systems. Using the usual notation $|i_1 i_2\rangle = |i_1\rangle|i_2\rangle$, the corresponding state superposition can be expressed as follows

$$|\psi\rangle = \sum_{i_1 i_2 = 0,1} a_{i_1 i_2}|i_1 i_2\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle, \tag{3.3}$$

where a_{ij} are complex numbers verifying the normalization condition

$$|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1 \tag{3.4}$$

describing the $\mathbf{CP^3}$ projective space. For n qubits, the general state has the following form

$$|\psi\rangle = \sum_{i_1 ... i_n = 0,1} a_{i_1 ... i_n}|i_1 ... i_n\rangle, \tag{3.5}$$

where a_{ij} satisfy the normalization condition

$$\sum_{i_1 ... i_n = 0,1} |a_{i_1 ... i_n}|^2 = 1. \tag{3.6}$$

This condition defines the $\mathbf{CP^{2^n - 1}}$ projective space generalizing the Bloch sphere associated with $n = 1$.

Roughly, the qubit systems can be represented by colored toric diagrams having a strong resemblance with a particular class of bosonic Adinkras, introduced in the study of supersymmetric

representation theory, by Gates and its group[22, 23, 24, 25, 26, 27, 28]. In fact, there are many kinds of such graphs. However, we consider a particular class called regular one consisting of 2^n vertices linked by n colored edges as will be shown latter on. An inspection, in graph theory of Adinkras and toric varieties, shows that we can propose the following correspondence connecting three different subjects

Adinkras	Colored Toric Geometry	Qubit systems
Vertices	Fixed points (vertices)	basis state
Number of colors	Number of toric actions (Dimension)	Number of qubits

Table 1: This table presents an one to one correspondence between colored toric geometry, Adinkras and qubit systems.

To see how this works in practice, we first change the usual toric geometry notation. Inspired by combinatorial formalism used in quantum information theory, the previous toric data can be rewritten as follows

$$\sum_{i_1 \ldots i_n = 0,1} q^a_{i_1 \ldots i_n} v_{i_1 \ldots i_n} = 0, \qquad a = 1, \ldots, r, \tag{3.7}$$

where the vertex subscripts indicate the corresponding quantum states. To illustrate this notation, we present a model associated with $\mathbf{CP^1} \times \mathbf{CP^1}$ toric variety. This model is related to $n = 2$ Adinkras with (1,2,1) data as listed in the classification. In this case, the combinatorial Mori vectors can take the following form

$$\begin{aligned} q^1_{i_1 i_2} &= (q^1_{00}, q^1_{01}, q^1_{10}, q^1_{11}) = (1, 0, 0, 1) \\ q^2_{i_1 i_2} &= (q^2_{00}, q^2_{01}, q^2_{10}, q^2_{11}) = (0, 1, 1, 0). \end{aligned} \tag{3.8}$$

The manifold corresponds to the toric equations

$$\sum_{i_1 i_2 = 0,1} q^a_{i_1 i_2} v_{i_1 i_2} = 0, \qquad a = 1, 2. \tag{3.9}$$

In colored toric geometry langauge, it is represented by 4 vertices $v_{i_1 i_2}$, belonging to Z^2, linked by four edges with two different colors c_1 and c_2. The toric data require the following vertices

$$v_{00} = (-1, 0), \ v_{01} = (0, 1), \ v_{10} = (0, -1), \ v_{11} = (1, 0) \tag{3.10}$$

with two colors. These data can be encoded in a toric graph describing two qubits and it is illustrated in figure 1.
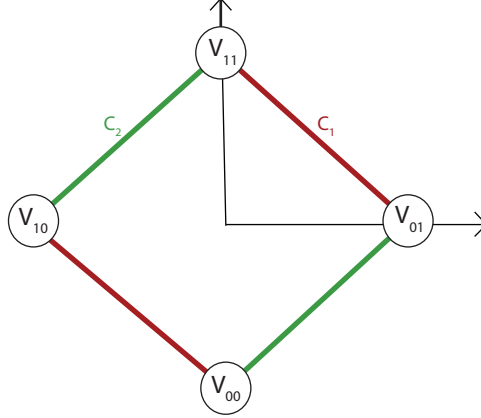
Figure 1: Toric Adinkra graph representation of for $n = 2$ qubits.

4   Quantum gates from geometry

Having examined the qubit object, we move now to build the quantum gates using colored toric geometry and Adinkra graph theory. The general study is beyond of the scope of this paper. We consider, however, lower dimensional cases. To do so, it is recalled that the classical gates can be obtained by combining Boolean operations as AND, OR,XOR, NOT and NAND. In fact, these operations act on input classical bits, taking two values 0 and 1, to produce new bits as output results. In quantum computation, gates are unitary operators in a 2^n dimensional Hilbert space. In connection with representation theory, they can be represented by $2^n \times 2^n$ matrix, belonging to $SU(2^n)$ Lie group, satisfying the following properties

$$U^+ = U^{-1}, \qquad det\, U = 1 \tag{4.1}$$

As in the classical case, there is an universal notation for the gates depending on the input qubit number. The latters are considered as building blocks for constructing circuits and transistors. For 1-qubit computation, the usual one is called NOT acting on the basis state as follows

$$|i_1\rangle \rightarrow |\overline{i_1}\rangle \tag{4.2}$$

In this toric geometry language, this operation corresponds to permuting the two toric vertices of $\mathbf{CP^1}$

$$\sigma : v_0 \leftrightarrow v_1 \tag{4.3}$$

This operation can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

6

which can be identified with U_{NOT} defining the NOT quantum gate. In this case, it is worth noting that the corresponding color operation is trivial since we have only one.

For 2-qubits, there are many universal gates. As mentioned previously, this system is associated with the toric geometry of $\mathbf{CP^1} \times \mathbf{CP^1}$. Unlike the 1-qubit case corresponding to $\mathbf{CP^1} \times \mathbf{CP^1}$, the quantum systems involve two different data namely the vertices and colors. Based on this observation, such data will produce two kinds of operations:

1. color actions

2. vertex actions.

In fact, these operations can produce CNOT and SWAP gates. To get such gates, we fix the color action according to Adinkra orders used in the corresponding notation. Following the colored toric realization of the 2-qubits, the color actions can be formulated as follows

$$
\begin{aligned}
c_1 &: \quad |i_1 i_2\rangle \to |\overline{i_1} i_2\rangle \\
c_2 &: \quad |i_1 i_2\rangle \to |i_1 \overline{i_2}\rangle.
\end{aligned}
\tag{4.5}
$$

In this color language, the CNOT gate

$$
CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\tag{4.6}
$$

can be obtained by using the following actions

$$
\begin{aligned}
c_1 &\to \quad c_1 \\
c_2 &\to \quad c_2 \otimes c_1.
\end{aligned}
\tag{4.7}
$$

A close inspection shows that the SWAP gate

$$
SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\tag{4.8}
$$

can be derived from the following permutation action

$$
c_1 \to c_2.
\tag{4.9}
$$

We expect that this analysis can be adopted to higher dimensional toric manifolds. For simplicity reason, we consider the geometry associated with TOFFOLI gate being an universal

gate acting on a 3-qubit. It is remarked that geometry can be identified with the blow up of $\mathbf{CP^1} \times \mathbf{CP^1} \times \mathbf{CP^1}$ toric manifold. In colored toric geometry language, this manifold is described by the following equations

$$\sum_{i_1 i_2 i_3 = 0,1} q^a_{i_1 i_2 i_3} v_{i_1 i_2 i_3} = 0, \qquad a = 1, \ldots, 5, \tag{4.10}$$

where 2^3 vertices $v_{i_1 i_2 i_3}$ belong to Z^3. They are connected by three different colors c_1, c_2 and c_3. These combinatorial equations can be solved by the following Mori vectors

$$\begin{aligned}
q^1_{i_1 i_2 i_3} &= (1,0,0,1,0,0,0,0) \\
q^2_{i_1 i_2 i_3} &= (0,1,0,0,1,0,0,0) \\
q^3_{i_1 i_2 i_3} &= (0,0,1,0,0,1,0,0) \\
q^4_{i_1 i_2 i_3} &= (1,-1,0,0,0,0,1,0) \\
q^5_{i_1 i_2 i_3} &= (0,0,1,1,0,0,0,1).
\end{aligned} \tag{4.11}$$

Thus, the corresponding vertices $v_{i_1 i_2 i_3}$ are given by

$$v_{000} = (1,0,0), \; v_{100} = (0,1,0), \; v_{010} = (0,0,1), \; v_{001} = (-1,0,0) \tag{4.12}$$

$$v_{110} = (0,-1,0), \; v_{101} = (0,0,-1), \; v_{110} = (-1,1,0), \; v_{111} = (0,-1,-1),$$

and they are connected with three colors. This representation can be illustrated in figure 2.
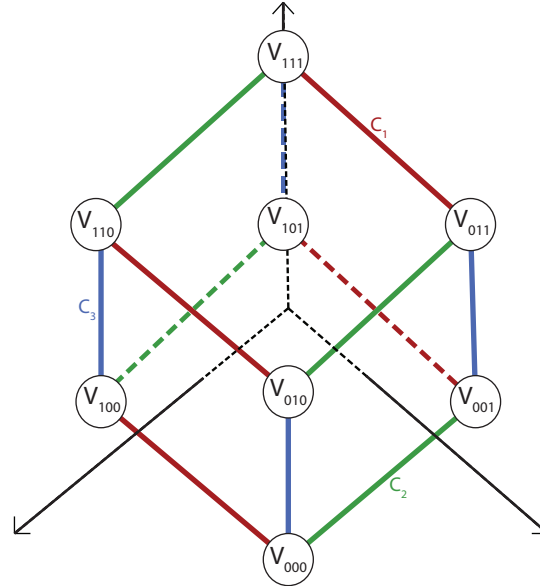

Figure 2: Regular Adinkra graphic representation for $n = 3$.

The TOFFOLI gate represented by $2^3 \times 2^3$ matrix

$$TOFFOLI = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.13}$$

can be obtained by the following color transformation

$$
\begin{aligned}
c_1 &\rightarrow c_1 \\
c_2 &\rightarrow c_2 \\
c_3 &\rightarrow c_3 \otimes c_2 \otimes c_1.
\end{aligned}
\tag{4.14}
$$

We expect that this analysis can be pushed further to deal with other toric varieties having non trivial Betti numbers.

5 Conclusion

Using toric geometry/ Adinkras correspondence, we have discussed qubit systems. More precisely, we have presented an one to one correspondence between three different subjects namely toric geometry, Adinkras and quantum information theory. We believe that this work may be explored to attack qubit system problems using geometry considered as a powerful tool to understand modern physics. In particular, we have considered in some details the cases of one, two and three qubits, and we find that they are associated with $\mathbf{CP^1}$, $\mathbf{CP^1} \times \mathbf{CP^1}$ and $\mathbf{CP^1} \times \mathbf{CP^1} \times \mathbf{CP^1}$ toric varieties respectively. Developing a geometric procedure referred to as colored toric geometry, we have revealed that the qubit physics can be converted into a scenario turning toric data of such manifolds by help of Adinkra graph theory. We have shown that operations on such data can produce universal quantum gates.

This work comes up with many open questions. A natural one is to examine super-projective spaces. We expect that this issue can be related to superqubit systems. Another question is to investigate the entanglement states in the context of toric geometry and its application including mirror symmetry. Instead of giving a speculation, we prefer to comeback these open questions in future.

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